Introduction to the Canonical Orlicz space $KS^{\theta}[\mathbb{R}_I^n]$

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Abstract. The objective of this paper is to construct Canonical Orlicz space and its fundamental properties. These spaces contains Henstock-Kurzweil integrable functions.

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1. Introduction and Preliminaries

In 2008 [14] Teeper.L Gill construct a new class of Banach space (Kuelb-Seadman space) $KS^p[\mathbb{R}^n]$ which parallels the standard L^p spaces but contains them as dense compact embeddings. KS^p contain the Henstock-Kurzweil integrable function and HK-measure, which generalizes the lebesgue measure.

After developed lebesgue integration theory by Hendri Lebesgue, Z.W. Birnbaum and W.Orlicz proposed a generalized space of L^p, so called later Orlicz space. In the old seminar text book (see [9] contains all the fundamental properties of Orlicz space with lebesgue measure. In [10] the theory of Orlicz space is in more general with Young function and the underlying measure. The basic ideas of the proofs of

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the theorem of Orlicz space are analogues of the basic results on L^p- space.
Henstock-integral was first developed by R. Henstock and J. Kurzweil independently during 1957-58 from Riemann integral with the concept of tagged partitions and gauge functions. Henstock integral is a kind of non absolute integral and contain lebesgue integral. See [13]. In [8] in sec 2.12, they developed an alternate approach of lebesgue measure on R^n. Recalling that lebesgue measure will leave geometric objects invariant under translations and rotations, and to assume that rotational and translation invariance is an intrinsic property of lebesgue measure.
In many many applications, rotational and translational invariance plays no role at all, it is the $\sigma-$finite nature of lebesgue measure that is critical.

Definition 1.1. [8] *We define* $\mathbb{R}_I^n = \mathbb{R}^n \times I_n$. *If* T *is a linear transformation on* R^n *and* $A_n = A \times I_n$, *we define* T_I *on* $\mathbb{R}_I^n$ *by* $T_I[A_n] = T[A]$, *we define* $B[\mathbb{R}_I^n]$ *to be the Borel* $\sigma-$*algebra for* $\mathbb{R}_I^n$, *where the topology for* $\mathbb{R}_I^n$ *is defined via the following class of open sets* $D_n = \{U \times I_n : U$ *is open in* $\mathbb{R}^n\}$. *For any* $A \in B[\mathbb{R}^n],\}$*we define* $\lambda_\infty(A_n)$ *on* $\mathbb{R}_I^n$ *by product measure* $\lambda_\infty(A_n) = \lambda_n(A) \times \Pi_{i=n+1}^\infty \lambda_I(I) = \lambda_n(A)$

Theorem 1.2. [2] $\lambda_\infty(.)$ *is a measure on* $B[\mathbb{R}_I^n]$ *is equivalent to n-dimensional Lebesgue measure on* R^n.

Corollary 1.3. *The measure* $\lambda_\infty(.)$ *is both translationally and rotationally invariant on* $(\mathbb{R}_I^n, B[\mathbb{R}_I^n])$ *for each* $n \in N$.

Thus we can construct a theory on $\mathbb{R}_I^n$ that completely paralleis that on $\mathbb{R}^n$.

Definition 1.4. [10] *A function* $\theta : \mathcal{R} \to \mathbb{R}^+$ *is said to be Young function so that* $\theta(x) = \theta(-x),\ \theta(0) = 0,\ \theta(x) \to \infty$ *as* $x \to \infty$ *but* $\theta(x_0) = +\infty$ *for some* $x_0 \in \mathcal{R}$ *is permitted.*

Definition 1.5. *Let $m : \mathbb{R}^+ \to \mathbb{R}^+$ be non decreasing right continuous and non negative function satisfying*

$$m(0) = 0, \ \textit{and} \ \lim_{t\to\infty} m(t) = \infty.$$

A function $M : \mathbb{R} \to \mathbb{R}$ is called an N-function if there is a function $'m'$ satisfying the above sense that

$$M(u) = \int_0^{|u|} m(t)dt.$$

Evidently, M is an N-function if it is continuous, convex, even satisfies

$$\lim_{u\to\infty} \frac{M(u)}{u} = \infty \ \textit{and} \ \lim_{u\to 0} \frac{M(u)}{u} = 0.$$

2. Purpose

In this paper, we want to study canonical Orlicz space. We want to investigate Canonical Orlicz space contains Orlicz space as dense embedding with compact support. We call $\mathcal{L}[\mathbb{R}_I^\infty]$ is canonical lebesgue space and $L[\mathbb{R}_I^\infty]$ is classical Orlicz space.

2.1. **Canonical $\mathcal{L}^1[\mathbb{R}_I^n]$.** We adopt the concept of [14], to use our $\mathcal{L}^1[R^n]$ in our work. Recalling their work

Let $\mathcal{B}_r$ is any closed cube of diagonal r centered at the origin in $\mathbb{R}^n$, with sides parallels to the coordinates axes and $\mathcal{E}_{\mathcal{B}_r}(x)$ is the characteristic function of $\mathcal{B}_r$. Let $\mathcal{E}_k(x)$ be the characteristic function of $\mathcal{B}_k$, so that $\mathcal{E}_k(x)$ is in $L^p[\mathbb{R}^n] \cap L^\infty[\mathbb{R}^n]$ for $1 \le p < \infty$. They define $F_k(.)$ on $L^1[\mathbb{R}^n]$ by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(x) f(x) dx$$

Fixing $t_k > 0$ and $\sum_{k=1}^\infty t_k = 1$ and

$$||f||_{L^1[\mathbb{R}^n]} = \sum_{k=1}^\infty [\int_{\mathbb{R}^n} \mathcal{E}_k(x) f(x) dx]$$

A carefully discussed in elementary analysis, the functions in $\mathcal{L}^1[R^n]$ are not uniquely defined. Let $L^1[\mathbb{R}^n]$ denote the set of equivalence classes of functions

in $\mathcal{L}^1[\mathbb{R}^n]$ that differ by a set of λ_n-measure zero. By a slight abuse, we will identify an integrable function f as measurable in $\mathcal{L}^1[\mathbb{R}^n]$ and its equivalence class in $L^1[\mathbb{R}^n]$.

Definition 2.1. *A measurable function $f : \Omega \to \mathbb{R}$ where $(\Omega, \Sigma, \lambda_\infty)$ is a measurable space is said to summable if the Lebesgue integral of the absolute value of f exists and finite,*

$$\sum_{k=1}^{\infty} \int_\Omega |\mathcal{E}_k(x) f(x) d\lambda_\infty| < \infty$$

Then we say $f \in \mathcal{L}^1[\Omega]$

Clearly Summable functions of $\mathcal{L}^1[\Omega]$ is Summable in $L^1[\Omega]$.

3. Canonical Orlicz class

Definition 3.1. [10] *Let $L^{-\theta}[\lambda_n]$ be the set of all $f : \Omega \to \mathbb{R}$, measurable for Σ, such that $\int_\Omega \theta(|f|) d\lambda_n \leq \infty$*

We start with an abstract measure space $(\Omega, \Sigma, \lambda_\infty)$ where Ω is some point set and Σ is a $\sigma-$ algebra of its subsets on which $\sigma-$ additive function $\lambda_\infty : \Sigma \to R^+$ is given.

To define Canonical Orlicz class we start with $L^{-\theta}[\lambda_n]$

Let f_n converges to f in $\mathcal{L}^{-\theta}[\lambda_n]$. Fixing $t_k > 0$ and $\sum_{k=1}^{\infty} t_k = 1$

Definition 3.2. *Let $KS^{-\theta}[\lambda_\infty]$ be the set of all $f_n : \Omega \to \mathbb{R}$, measurable for Σ such that*

$$\sum_{k=1}^{\infty} t_k [\int_\Omega \mathcal{E}_k(x) \theta(|f|) d\lambda_\infty] < \infty$$

where $\{f_n\} \to f$ is fundamental sequence in $\mathcal{L}^{-\theta}[\lambda_n]$

We call $KS^{-\theta}[\lambda_\infty]$ be Canonical Orlicz class. To avoid some annoying repetition, we assume our measure λ_∞ have finite subset property. That is Let $(\Omega, \Sigma, \lambda_\infty)$ be a measure space of infinite measure. We say that this measure space has the finite subset property property if for every $E \in \Sigma$ with $\lambda_\infty[E] = \infty$, there exists a

family of subsets $\{E_i\}_{i=1}^{\infty} \subset \Sigma$ with $E_i \subset E$; $\lambda_\infty[E_i] < \infty$ and $\lambda_\infty[\bigcup_{i=1}^{\infty} E_i] = \infty$ If$KS^{-\theta}[\lambda_\infty]$ is completion of $\mathcal{L}^{-\theta}[\lambda_n]$, then following result holds.

Theorem 3.3. *$\mathcal{L}^{-\theta}[\lambda_n] \subset KS^{-\theta}[\lambda_\infty]$ as dense.*

Proof. Let $f \in \mathcal{L}^{-\theta}[\mathbb{R}^n]$ then clearly $f \in KS^{-\theta}[\mathbb{R}^n]$ □

All bounded functions, but not all summable functions, belong to the class $KS^{-\theta}$. It is easy to show that every function in the class $KS^{-\theta}$. is summable.

Theorem 3.4. *Every function $u(x)$ which is summable on $\mathcal{G}$, where $\mathcal{G}$ is a bounded closed set in a finite-dimensional Euclidean space is in some Canonical Orlicz class.*

Proof. Let $\mathcal{G}_n = \mathcal{G}\{k-1 \leq |u(x)| < k\}$
Then

$$\sum_{k=1}^{\infty} t_k \mu(\mathcal{G}_n) \leq \int_{\mathcal{G}} |u(x)| d\lambda_\infty + \mu(\mathcal{G}_n) < \infty$$

where $\mu(\mathcal{G}_n)$ is mesh of $\mathcal{G}_n$.
Let us construct an indefinitely increasing sequence $\{\alpha_k\}$ such that

$$\sum_{k=1}^{\infty} \alpha_k t_k \mu(\mathcal{G}_n) < \infty \tag{1}$$

Let

$$\mathcal{P}_k(t) = \begin{cases} t \ \text{ for } 0 \leq t < 1; \\ \alpha_k \ \text{for } k \leq t < k+1 \end{cases}$$

where $k = 1, 2, ..$ The function $\mathcal{P}(t)$ possesses all the properties required in order that

$$M(u) = \int_0^{|u|} \mathcal{P}(t) dt$$

be an N-function. Since

$$M(k) = \int_0^k \mathcal{P}(t) dt \leq \sum_{k=1}^{\infty} \alpha_k t_k$$

By (1)

$$\begin{aligned}\int_{\mathcal{G}} M[u(x)]d\lambda_{\infty} &= \sum_{k=1}^{\infty}\int_{\mathcal{G}_n} M[u(x)]d\lambda_{\infty} \\ &\leq \sum_{k=1}^{\infty} M(t)\mu(\mathcal{G}_n) \\ &\leq \sum_{k=1}^{\infty} \alpha_k t_k \mu(\mathcal{G}_n) \\ &< \infty\end{aligned}$$

So, $u(x) \in KS^{-\theta}[\mathcal{G}]$ □

Theorem 3.5. *The Jensen integral inequality: Let $f(x) \in KS^{-\theta}$; then the inequality*

$$\sum_{k=1}^{\infty} t_k \theta\{\frac{\int_{\mathcal{G}} \mathcal{E}_k f(x)dx}{\mu(\mathcal{G})}\} \leq \sum_{k=1}^{\infty} t_k \frac{\int_{\mathcal{G}} \mathcal{E}_k \theta(f(x))dx}{\mu(\mathcal{G})}$$

holds, this inequality will called the Jensen integral inequality.

Proof. Let $f(x)$ is a continuous function. Suppose $\epsilon > 0$ is an arbitary prescribed number. The set $\mathcal{G}$ can be decomposed into k subsets $\mathcal{G}_i$ such that $\mu(\mathcal{G}_i) = \mu(\mathcal{G})/k,\ (i = 1, 2, ..., k)$,

$$|\theta(\int_{\mathcal{G}} \frac{\mathcal{E}_k f(x)}{\mu(\mathcal{G})}dx) - \theta(\sum_{i=1}^{\infty} \mathcal{E}_k f(x_i)\frac{1}{k})| < \epsilon$$

Implies

$$\sum_{k=1}^{\infty} t_k |\theta(\int_{\mathcal{G}} \frac{\mathcal{E}_k(x) f(x)}{\mu(\mathcal{G})}dx) - \mathcal{E}_k \theta(\sum_{i=1}^{\infty} f(x_i)\frac{1}{k})| < \epsilon$$

where x_i is some point in the set $\mathcal{G}_i$

As $\theta(\frac{f_1+f_2+..+f_k}{k}) \leq \frac{1}{k}[\theta(f_1) + \theta(f_2) + .. + \theta(f_k)]$

So,

$$\begin{aligned}\sum_{k=1}^{\infty} t_k \theta(\frac{\int_{\mathcal{G}} \mathcal{E}_k(x)\theta(f(x))dx}{\mu(\mathcal{G})}) &\leq (\sum_{i=1}^{\infty} \mathcal{E}_k(x) f(x_i)\frac{1}{k}) + \epsilon \\ &\leq \sum_{i=1}^{\infty} \frac{\theta(f(x_i)\mathcal{E}_k(x_i)}{k} + \epsilon \\ &\leq \frac{\int_{\mathcal{G}} \mathcal{E}_k(x)\theta(f(x_i))dx}{\mu(\mathcal{G})}\end{aligned}$$

□

Theorem 3.6. *The space $KS^{-\theta}[\lambda_\infty]$ is absolutely convex. i.e. if $f, g \in Ks^{-\theta}[\lambda_\infty]$ and α, β are scalars such that $|\alpha| + |\beta| \leq 1$ then $\alpha f + \beta g \in KS^{-\theta}[\lambda_\infty]$. Also $h \in KS^{-\theta}[\lambda_\infty]$, $|f| \leq |h|$, f is measurable implies $f \in KS^{-\theta}[\lambda_\infty]$*

Proof. Let $f, g \in KS^{-\theta}[\lambda_\infty]$, then by monotonicity and convexity of θ, we get $0 < \gamma = |\alpha| + |\beta| \leq 1$

And

$$\begin{aligned}
\sum_{k=1}^{\infty} t_k \int_\Omega \mathcal{E}_k(x)\theta(|\alpha f + \beta g|)d\lambda_\infty &\leq \sum_{k=1}^{\infty} t_k \int_\Omega \mathcal{E}_k(x)\theta(|\alpha||f|)d\lambda_\infty + \sum_{k=1}^{\infty} t_k \int_\Omega \mathcal{E}_k(x)\theta(|\beta||g|)d\lambda_\infty \\
&\leq |\alpha| \sum_{k=1}^{\infty} t_k \int_\Omega \mathcal{E}_k(x)\theta(|f|)d\lambda_\infty + |\beta| \sum_{k=1}^{\infty} t_k \int_\Omega \mathcal{E}_k(x)\theta(|g|)d\lambda_\infty \\
&< \infty
\end{aligned}$$

So, $\alpha f + \beta g \in KS^{-\theta}[\lambda_\infty]$

For the second part $|f| \leq |h|$

Then

$$\sum_{k=1}^{\infty} t_k \int_\Omega \mathcal{E}_k(x)\theta(|f|)d\lambda_\infty \leq \sum_{k=1}^{\infty} t_k [\int_\Omega \mathcal{E}_k(x)\theta(|h|)d\lambda_\infty$$

So, $\sum_{k=1}^{\infty} t_k \int_\Omega \mathcal{E}_k(x)\theta(|f|)d\lambda_\infty < \infty$

so, $f \in KS^{-\theta}[\lambda_\infty]$ □

4. Conclusion

In this paper we have found Canonical Orlicz class contains Orlicz class as dense. Orlicz class is convex. In next paper we come to Canonical Orlicz space in details.

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